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CONTACT PROBLEM FOR A RING PLATE ON AN ELASTIC HALF-SPACE. VARIATIONAL APPROACH

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A contact problem of an axisymmetrically loaded flexible ring plate lying frictionlessly on an elastic half-space is considered. The plate subsidences are represented as a power series with unknown coefficients, which are determined by the Rayleigh–Ritz method using the minimum condition for the total strain energy of the plate and the elastic foundation. The method of orthogonal polynomials is used implicitly.

The problem of axisymmetrical bending of a transversely loaded flexible ring plate which lies frictionlessly on an elastic half-space with distributional properties has not been adequately studied. This problem for a ring punch was studied in detail by Aleksandrov in [1–3], where asymptotic solutions were obtained for narrow and wide ring punches. In the present paper, to solve this problem a variational approach is used as is done in [4] to solve a contact problem for a circular plate on an elastic half-space. However, in contrast to [4], the static boundary conditions at the edges of a simply lying plate are not satisfied. They are known [5] to be satisfied automatically by taking a sufficiently large number of terms of a series approximating the subsidence function of a ring plate. The above approach was used in [6] to obtain a solution of the contact problem of a beam on an elastic half-plane.

We consider a ring plate lying frictionlessly on an elastic homogeneous isotropic half-space having constant modulus of elasticity E and Poisson ratio ν and subjected to a transverse axisymmerical load $q(r)$ (Fig. 1). We shall seek the plate subsidences and the reactive stress distribution in the contact zone between the plate and the elastic half-space caused by the applied load. In calculations, we use the following assumptions:

(a) tangential stresses do not occur between the plate and the foundation during bending;

(b) the plate thickness is small and the Kirchhoff–Love hypotheses are valid;

(c) compressive and tensile stresses can occur between the plate and the foundation.

We express the deflection function of the ring plate in the form of the series

$$w(r) = \sum_{m=0}^{\infty} A_m \frac{r^m}{b^m}, \qquad (1)$$

where b is the external radius of the plate and A_m are the desired coefficients. The strain energy of the ring plate in bending can be written in the form [7]

$$U = \pi D \int_a^b \Big[\Big(\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\Big)^2 - 2(1-\nu_p)\frac{1}{r}\frac{dw}{dr}\frac{d^2w}{dr^2}\Big] r\,dr$$

$$= \frac{\pi D}{b^2}\Big[A_1^2 \ln\frac{b}{a} + \sum_{m=2}^{\infty} \frac{m^4 - 2(1-\nu_p)m^2(m-1)}{2m-2}\Big(1 - \frac{a^{2m-2}}{b^{2m-2}}\Big)A_m^2$$

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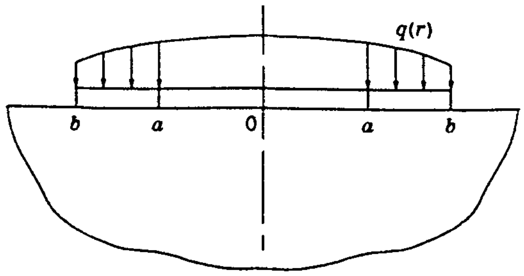

Fig. 1

$$+2\sum_{\substack{m=2\\m\neq n}}^{\infty}\sum_{n=1}^{\infty}\frac{m^2n^2-(1-\nu_p)mn(m-1)}{m+n-2}\left(1-\frac{a^{m+n-2}}{b^{m+n-2}}\right)A_mA_n\Big],$$

where ν_p is the Poisson ratio of the plate material and a is the internal radius of the ring plate.

The strain compatibility condition implies that the plate deflections $w(r)$ are equal to the subsidences $v(r)$ of the elastic half-space

$$w(r)=v(r)=2\,\frac{1-\nu^2}{\pi E}\int_0^{\pi}\int_a^b p(\rho)\,\frac{\rho\,d\rho\,d\varphi}{(r^2+\rho^2-2r\rho\cos\varphi)^{1/2}},$$

where $p(r)$ is the distribution of the contact stresses occurring between the plate and the elastic foundation because of plate bending.

After integrating over the variable φ, we have

$$v(r)=4\,\frac{1-\nu^2}{\pi E}\int_a^b p(\rho)K\Big(\frac{2\sqrt{\rho r}}{r+\rho}\Big)\,\frac{\rho\,d\rho}{\rho+r},$$

where $K(z)$ is the complete elliptic integral [8].

To determine a contact-stress distribution that causes deflections of the ring plate in the form (1), we solve the auxiliary integral equation

$$\sum_{m=0}^{\infty}A_m\,\frac{r^m}{b^m}=4\,\frac{1-\nu^2}{\pi E}\int_a^b \rho p(\rho)K\Big(\frac{2\sqrt{\rho r}}{\rho+r}\Big)\,\frac{d\rho}{\rho+r}. \tag{2}$$

In (2), we introduce new variables [1]

$$\rho=a\,\exp\left(\frac{1+x}{\lambda}\right),\quad r=a\,\exp\left(\frac{1+t}{\lambda}\right),\quad \lambda=\ln\left(\frac{2}{b/a}\right)$$

and reduce the integral equation (2) to the form

$$\int_{-1}^{1}\varphi(x)K\Big[\Big(\mathrm{ch}\,\frac{t-x}{2\lambda}\Big)^{-1}\Big]\Big/\cosh\frac{t-x}{2\lambda}\,dx=\frac{\pi E\lambda}{2a(1-\nu^2)}\,(r/a)^{1/2}\sum_{m=0}^{\infty}A_m\,\frac{r^m}{b^m},$$

$$\varphi(x)=(\rho/a)^{3/2}p(\rho). \tag{3}$$

Taking account of the logarithmic singularity in the kernel of the integral equation (3), similarly to [3] we assume that

$$\Big(\cosh\frac{t-x}{2\lambda}\Big)^{-1}K\Big[\Big(\cosh\frac{t-x}{2\lambda}\Big)^{-1}\Big]=-\ln\Big|\frac{t-x}{\lambda}\Big|+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}C_{mn}(\lambda)T(x)T(t),$$

$$C_{mn}(\lambda) = \beta_{mn} \int_{-1}^{1}\int_{-1}^{1} \left[\frac{K(1/\cosh((t-x)/2\lambda))}{\cosh((t-x)/2\lambda)} + \ln\left|\frac{t-x}{\lambda}\right|\right] \frac{T_m(x)T_n(t)\,dx\,dt}{[(1-x^2)(1-t^2)]^{1/2}}, \tag{4}$$

$$\beta_{00} = 1/\pi^2, \quad \beta_{0m} = \beta_{m0} = \beta_{0n} = \beta_{n0} = 2/\pi^2, \quad \beta_{mn} = \beta_{nm} = 4/\pi^2,$$

where $T_m(z)$ is a Chebyshev polynomial of the first kind [8].

The representation (4) enables us to seek a solution of the integral equation (3) in the form of a series in Chebyshev polynomials of the first kind

$$\varphi(x) = (1-x^2)^{-1/2} \sum_{k=0}^{\infty} B_k T_k(x), \tag{5}$$

where B_k are the desired coefficients.

Then we use the spectral [2]

$$\int_{-1}^{1} \ln|t-x| T_m(x)(1-x^2)^{-1/2}\,dx = \begin{cases} -\pi\ln 2, & m=0, \quad |x<1|, \\ -\pi/mT_m(t), & m=1,2,\ldots \end{cases}$$

and orthogonality relations for the chosen Chebyshev polynomials [8]:

$$\int_{-1}^{1} T_m(x)T_n(x)(1-x^2)^{-1/2}\,dx = \begin{cases} 0, & m \neq n, \\ \pi, & m=n=0, \\ \pi/2, & m=n. \end{cases}$$

As a result, we have an equation that relates the coefficients of expansions (1) and (5). We multiply both sides of this equation by $T_i(t)(1-t^2)^{-1/2}\,dt$, integrate over the interval $[-1,+1]$, and obtain the infinite system

$$[\alpha][B] = [A], \tag{6}$$

$$[\alpha] = \begin{vmatrix} \ln 2\lambda + C_{00} & C_{10}/2 & C_{20}/2 & C_{30}/2 & \ldots \\ C_{01}/2 & 1/2 + C_{11}/4 & C_{21}/4 & C_{31}/4 & \ldots \\ C_{02}/2 & C_{12}/4 & 1/4 + C_{22}/4 & C_{32}/4 & \ldots \\ C_{03}/2 & C_{13}/4 & C_{23}/4 & 1/6 + C_{33}/4 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}, \quad [B] = \begin{vmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \\ \ldots \end{vmatrix},$$

$$[A] = \frac{E\lambda e^{1/\lambda}}{2a(1-\nu^2)} \begin{vmatrix} I_0(0,5/\lambda)a/b\,e^{1/2\lambda} & I_0(1,5/\lambda)a^2/b^2e^{3/2\lambda} & I_0(2,5/\lambda)\ldots \\ I_1(0,5/\lambda)a/b\,e^{1/2\lambda} & I_1(1,5/\lambda)a^2/b^2e^{3/2\lambda} & I_1(2,5/\lambda)\ldots \\ I_2(0,5/\lambda)a/b\,e^{1/2\lambda} & I_2(1,5/\lambda)a^2/b^2e^{3/2\lambda} & I_2(2,5/\lambda)\ldots \\ I_3(0,5/\lambda)a/b\,e^{1/2\lambda} & I_3(1,5/\lambda)a^2/b^2e^{3/2\lambda} & I_3(2,5/\lambda)\ldots \\ \ldots & \ldots & \ldots \end{vmatrix} \begin{vmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \\ \ldots \end{vmatrix},$$

where $I_k(z)$ is the modified Bessel function [8] and the nondiagonal coefficients of the matrix $[\alpha]$ are significantly less than the diagonal elements.

We write the solution of system (6) in matrix form:

$$[B] = [\alpha]^{-1}[A], \tag{7}$$

$$[\alpha]^{-1} = \begin{vmatrix} \alpha_{00} & \alpha_{01} & \alpha_{02} & \alpha_{03} & \alpha_{04} & \ldots \\ \alpha_{10} & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \ldots \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} & \ldots \\ \alpha_{30} & \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}.$$

The representation (7) enables us to express the coefficients B_n of series (5) in terms of the coefficients A_m of series (1). This, in turn, makes it possible to express the energy of the elastic foundation as a quadratic function of the coefficients A_m. Indeed, we find the work of the reactive stresses $p(r)$ in displacements $w(r)$:

$$T = \pi \int_a^b p(r)w(r)r\,dr = \frac{\pi^2 a^2 e^{2/\lambda}}{\lambda} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} e^{m/\lambda} \frac{a^m}{b^m} A_m B_k I_k\Big(\frac{m+2}{\lambda}\Big).$$

The work of the external forces Π done in the displacement $w(r)$ during bending of the plate can be presented in the form

$$\Pi = -2\pi \int_a^b q(r)w(r)r\,dr = -2\pi \sum_{m=0}^{\infty} A_m b^{-m} \int_a^b q(r) r^{m+1}\,dr.$$

The resulting expression for the total potential energy of the system comprising the plate and the foundation can be written as follows:

$$U = \frac{\pi D}{b^2} \Big[A_1^2 \ln\frac{b}{a} + \sum_{m=2}^{\infty} \frac{m^4 - 2(1-\nu_p)m^2(m-1)}{2m-2} \Big(1 - \frac{a^{2m-2}}{b^{2m-2}}\Big) A_m^2$$
$$+ 2 \sum_{\substack{m=2\\ m\neq n}}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n^2 - (1-\nu_p)mn(m-1)}{m+n-2} \Big(1 - \frac{a^{m+n-2}}{b^{m+n-2}}\Big) A_m A_n$$
$$+ \frac{\pi^2 E a}{2(1-\nu^2)} e^{2.5/\lambda} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} e^{m/\lambda} \frac{a^m}{b^m} I_k\Big(\frac{m+2}{\lambda}\Big) \alpha_{k,n} A_m A_n \Big] - 2\pi \sum_{m=0}^{\infty} A_m b^{-m} \int_a^b q(r) r^{m+1}\,dr. \qquad (8)$$

We differentiate (8) with respect to A_i ($i = 0, 1, 2, \ldots$) and arrive at the infinite set of linear algebraic equations for the desired A_i:

$$\sum_{n=0}^{\infty} \delta_{mn} A_n + \Delta_m = 0, \quad m = 0, 1, 2, \ldots,$$
$$\delta_{0n} = \delta_{n0} = \beta \frac{\pi^2 e^{5/2\lambda} a}{2b} \Big[\sum_{k=0}^{\infty} \alpha_{kn} I_k\Big(\frac{2}{\lambda}\Big) + e^{n/\lambda} \frac{a^n}{b^n} \sum_{k=0}^{\infty} \alpha_{k0} I_k\Big(\frac{2+n}{\lambda}\Big)\Big],$$
$$\delta_{11} = 2\beta \ln\frac{b}{a} + 2\beta \frac{\pi^2 e^{5/2\lambda} a}{2b} \frac{a}{b} \sum_{k=0}^{\infty} \alpha_{k1} I_k\Big(\frac{3}{\lambda}\Big), \qquad (9)$$
$$\delta_{1n} = \delta_{n1} = 2 \frac{n^2 - (1-\nu_p)n(n-1)}{n-1} \Big(1 - \frac{a^{n-1}}{b^{n-1}}\Big)$$
$$+ \beta \frac{\pi^2 e^{5/2\lambda} a}{2b} \Big[\frac{a}{b} \sum_{k=0}^{\infty} \alpha_{k1} I_k\Big(\frac{3}{\lambda}\Big) + \frac{a^n}{b^n} e^{n/\lambda} \sum_{k=0}^{\infty} \alpha_{k1} I_k\Big(\frac{2+n}{\lambda}\Big)\Big],$$
$$n = 2, 3, 4, \ldots,$$
$$\delta_{mn} = \delta_{nm} = 2 \frac{m^2 n^2 - 2(1-\nu_p)mn(m-1)}{m+n+2} \Big(1 - \frac{a^{m+n-2}}{b^{m+n-2}}\Big)$$
$$+ \beta \frac{\pi^2 e^{5/2\lambda} a}{2b} \Big[\frac{a^m}{b^m} e^{n/\lambda} \sum_{k=0}^{\infty} \alpha_{km} I_k\Big(\frac{n+2}{\lambda}\Big) + \frac{a^n}{b^n} e^{m/\lambda} \sum_{k=0}^{\infty} \alpha_{kn} I_k\Big(\frac{m+2}{\lambda}\Big)\Big],$$
$$\Delta_m = -\frac{2}{b^{m-2} D} \int_a^b q(r) r^{m+1}\,dr.$$

Here $\beta = Eb^3/(\pi D(1-\nu^2))$ is the flexibility factor given by Gorbunov-Posadov [9].

TABLE 1

Expansion Coefficients (1) *for* $\nu = 1/6$

a/b	A_0	A_1	A_2	A_3	A_4	A_5	A_6
				$\beta = 0.01$			
0.9	11.24664	−0.00919	−0.00009	−0.00003	—	—	—
0.3	11.09959	−0.09326	−0.00458	−0.00234	−0.00109	−0.00037	0.00007
				$\beta = 20.0$			
0.9	0.00962	−0.00414	−0.00003	−0.00001	—	—	—
0.3	0.02258	−0.01924	0.00136	0.00058	0.00018	0.00001	−0.00004

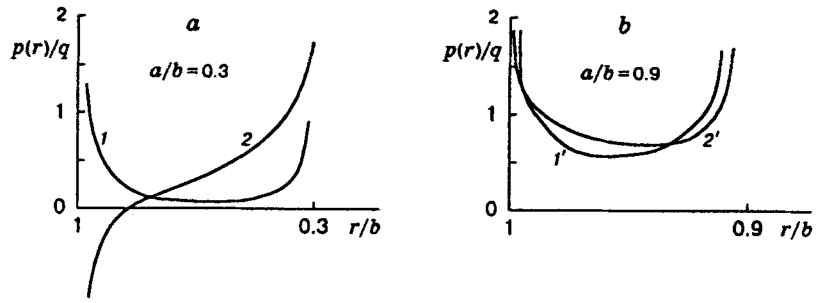

Fig. 2

Once the coefficients A_m (m = 0, 1, 2, ...) are determined from Eq. (9), we can find B_k, where $k = 0, 1, 2, 3, \ldots$, using formula (7); this makes it possible to determine the contact-stress distribution under the ring plate by means of formula (5):

$$p(r) = \frac{a^{1.5}}{\lambda r^{1.5}} [\ln(r/a)\ln(b/r)]^{-1/2} \sum_{m=0}^{\infty} B_m T_m(\lambda \ln(r/a) - 1).$$

Using known formulas [7], one can calculate the forces in the plate. The approach considered was used to calculate ring plates with various ratios a/b and flexibility factors β. The coefficients C_{mn} (4) were determined numerically using quadrature formula [10, formula (25.4.38)]. In calculations, the first 10 terms were retained in series (1) and (5) and the expressions for δ_{mn} (9). The best convergence of solution was observed for narrow plates ($a/b \approx 1$), where only three terms of series (1) and (5) are sufficient for practical calculations. This conclusion is supported by the data in Table 1, where the coefficients of series (1) are given for two values of a/b for a uniformly loaded plate ($\beta = 0.01$ and 20.0). Figure 2 shows the diagrams of the reactive stresses for these flexibility factors (curves 1 and 1′ refer to $\beta = 0.01$, and curves 2 and 2′ to $\beta = 20$) and the values of a/b. It can be seen that the stress distribution for a narrow plate (Fig. 2b) depends weakly on the flexibility factor β. For wide plates (Fig. 2a), it depends significantly on β and it is, therefore, necessary to preserve five or six terms in series (1) and (5).

One can also see in Fig. 2 that, for a ring plate with small a/b, tensile contact stresses at the external edge can occur for large values of β. To avoid this, one should decrease the value of β by increasing the flexural rigidity of the plate or by decreasing the modulus of elasticity of the foundation. For example, for a ring plate having $a/b = 0.3$ with $\beta = 4.526$ and uniformly distributed load, the tensile contact stresses occur in the proximity of its external edge.

For rigid ring plates, the author's results coincide with the asymptotic solutions obtained by Aleksandrov [1].

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